20/13

THE
MATHEMATICAL GAZETTE.

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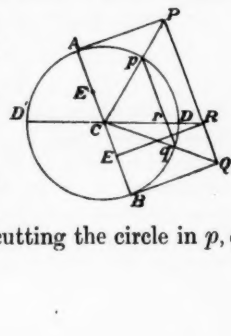
A CONIC CAN BE DRAWN THROUGH ANY FIVE POINTS.

WITH the aid of the construction (I.) given below, I am able to give an elementary geometrical proof of the above theorem, analogous to the algebraical proof, viz., by reduction to centre and principal axes. It is customary to assume, without proof, in geometrical conics that the curves represented algebraically by $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$, and $y^2 = px$ are conics whether the axes be rectangular or not. It is quite legitimate when the axes are rectangular, as we can construct vertices, foci, and eccentricity; but when they are not, we ought to show how to reduce the curves to their principal axes, which I have done in II. and III. In the latter part of the proof I have followed the order and method adopted in Dr. Macaulay's *Geometrical Conics*, p. 202, though the proof might have been slightly shortened by use of the cross-ratio property of conics.

I. CONSTRUCTION. *To place a chord in a circle, parallel to one diameter, and divided by another internally or externally in a given ratio* μ.



To place in the circle ADB, centre C, a chord parallel to AB, and divided by CD in a given ratio μ.

Divide AB, internally or externally as the case may be, at E in the ratio

$$EA : EB = \mu.$$

At A, B, E draw perpendiculars AP, BQ, ER, the latter meeting CD in R; draw RPQ parallel to AB; join CP, CQ, cutting the circle in p, q; join pq cutting DC in r.

Then evidently $CP=CQ$, and $Cp=Cq$;

$$\therefore\ pq \text{ is parallel to } PQ;$$

$$\therefore\ rp : rq = RP : RQ = EA : EB = \mu.$$

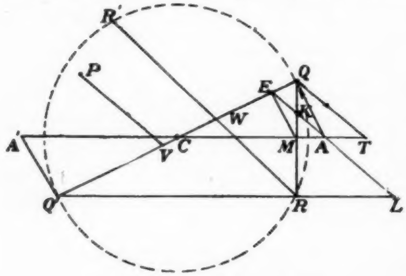
Again, no other chord $p'q'r'$ can be drawn divided in the same ratio in the same order; for since pqr, $p'q'r'$ would be similarly divided, pp', qq', rr' would be concurrent; but pp' qq' meet on the diameter at right angles to AB, since they join the extremities of chords parallel to AB; and r, r' are on a different diameter.

It is evident, however, that if we divide AB at E' so that $E'B : E'A = EA : EB = \mu$, we shall get, on the other side of AB, a second chord, similarly divided, but with corresponding points in reversed order.

Also, when the diameters CD and AB are at right angles, the construction gives chords of zero length at D, D', all chords of finite length parallel to AB being in this case, of course, bisected by CD.

In what follows, all equalities are true in sign as well as in magnitude, and all constructions are real.

II. THEOREM. *The curve $PV^2 : VQ\,.\,VQ' = \mu$, a constant in magnitude and sign, QQ' being fixed real points, and PV parallel to a fixed direction meeting QQ' in V, is a hyperbola or an ellipse, according as μ is positive or negative.*



Let P be a point on the curve, PV parallel to the fixed direction. Bisect QQ' at C; with centre C draw the circle QRQ', and place in it, on the Q side of C, a chord RWR' parallel to PV, and divided by QQ' (internally or externally as μ is negative or positive), so that $WR : WR' = \mu$;

$$\therefore\ \frac{RW^2}{WQ\,.\,WQ'} = \frac{WR\,.\,WR}{WR\,.\,WR'}, \text{ since } QRQ'R' \text{ are concyclic,}$$

$$= \mu;$$

$$\therefore\ R \text{ is a point on the curve.}$$

Draw QT, CT parallel to PV, $Q'R$.

Then since $Q'RQ$, in a semicircle, is a right angle, CT bisects QR at right angles in M.

Determine A, A' so that $CM\,.\,CT$* $= CA^2 = CA'^2$.

* The points M, T are on the same side of C, for the parallels WR, QT are on the same side of C, by hypothesis, and M is midway between the parallels.

Join AQ, and draw $EKAL$ parallel to PV, cutting QQ, QR, $Q'R$ in E, K, L; and join EM.

Then $$\frac{CM}{CA}=\frac{CA}{CT}=\frac{CE}{CQ}, \text{ by similar triangles,}$$

$$\therefore\ EM \text{ is parallel to } QA;$$

$$\therefore \text{ by similar triangles, } \frac{EK}{EA}=\frac{EM}{EM+QA}=\frac{CE}{CE+CQ}$$

$$=\frac{CE}{Q'E}=\frac{EA}{EL};$$

$$\therefore\ EK\,.\,EL=EA^2=AE^2;$$

$$\therefore\ \frac{AE^2}{EQ\,.\,EQ'}=\frac{EK\,.\,EL}{EQ\,.\,EQ'}=\frac{WR\,.\,WR}{WQ\,.\,WQ'}=\mu;$$

$\therefore$ A and also A' are points on the curve.

Construct the conic with major axis AA', and semi-minor axis CB given by the relation

$$\frac{BC^2}{AC^2}=-\frac{QM^2}{MA\,.\,MA'}.$$

This passes through Q, and since $CM\,.\,CT=CA^2$, QT is the tangent at Q, and therefore conjugate in direction to CQ.

Therefore all points P on this conic satisfy the relation

$$\frac{PV^2}{VQ\,.\,VQ'}=\frac{AE^2}{EQ\,.\,EQ'}=\mu;$$

and other points P, on the same side of the tangents at QQ' as the conic, will have a different value of PV^2 for the same value of $VQ\,.\,VQ'$; and all points on the other side of these tangents will have a different sign for $PV^2 : VQ\,.\,VQ'$;

$\therefore$ the curve and the conic are identical.

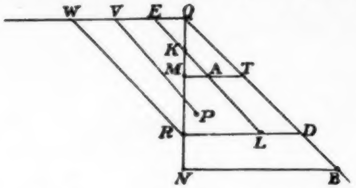
If PV, QQ' are originally at right angles, the conic through P with QQ' as major axis, and that alone, coincides with the curve.

Lastly, the conic is a hyperbola or an ellipse according as M is without or within AA', i.e. according as E is without or within QQ', since AQ, ME, $A'Q'$ are parallel, i.e. according as $\mu\left(=\dfrac{AE^2}{EQ\,.\,EQ'}\right)$ is positive or negative.

III. THEOREM. *The curve $PV^2=\mu\,.\,QV$, where μ is a constant positive length, and PV, parallel to a fixed direction, meets a fixed line through a fixed point Q in V, is one or other of two parabolas, according to the direction of QV which is considered positive.*

Draw QN at right angles to QV, and $QB=\mu$ parallel to PV, in the direction such that the perpendicular BN upon QN is positive.

Make $QD=BN$, along QB; draw DR perpendicular to QN; bisect QR at right angles by MT; bisect MT at A; and draw $EKAL$ and RW each parallel to PV.



Then

$$\frac{RW}{QW}=\frac{DQ}{DR}=\frac{BQ}{BN}=\frac{QB}{DQ}=\frac{\mu}{RW};$$

$$\therefore \quad RW^2=\mu \,.\, QW,$$

and hence R is on the curve.

Again, $\quad EQ=AT=MA$;

$\therefore \quad EK=KA$, and $EA=AL$;

$\therefore \quad EK\,.\,EL=EA^2=AE^2$;

$$\therefore \quad \frac{AE^2}{RW^2}=\frac{EK\,.\,EL}{RW\,.\,RW}=\frac{KE}{RW}=\frac{QE}{QW};$$

$\therefore \quad AE^2=\mu\,.\,QE$, and hence A is on the curve.

Draw the parabola with vertex A, axis AM, and latus rectum l, given by $QM^2=l\,.\,AM$.

This passes through Q, and since $MA=AT$, QT is the tangent at Q, hence all points P on the parabola satisfy the relation

$$\frac{PV^2}{AE^2}=\frac{QV}{QE}, \text{ or } PV^2=\mu\,.\,QV,$$

i.e. all points P on this parabola, and as in the previous case no points not on it, are on the curve;

$\therefore$ the curve and the parabola are identical.

If QV be measured in the opposite sense, then measuring off QB also in the opposite sense, and proceeding as before, we shall get another parabola equal to the first but lying on the other side of the tangent QT.

IV. Construction. *Three points C, V, W being in a line, to determine on the line two others Q, Q' equidistant from C, such that $VQ\,.\,VQ' : WQ\,.\,WQ' = a^2 : b^2$, a and b being given lines.*

Firstly. CVW being in the order written, determine the unique point M, which will be without VW, so that

$$MV : MW = a^2 : b^2.$$

Determine the real points K (between V and W) and L such that

$MV\,.\,MW=MK^2=ML^2$, M bisecting KL. ....(1)

Determine Q, Q' so that

$$CQ^2=CQ'^2=CL\,.\,CK;$$

then $$CQ^2 = CM^2 - ML^2 = CM^2 - MV \,.\, MW;$$
and
$$\begin{aligned} VQ \,.\, VQ' &= CV^2 - CQ^2 = CV^2 - CM^2 + MV \,.\, MW \\ &= MV(CV + CM) + MV \,.\, MW \\ &= MV(CV + CW). \end{aligned}$$

Similarly $WQ \,.\, WQ' = MW(CV + CW)$;

$$\therefore\ VQ \,.\, VQ' : WQ \,.\, WQ' = MV : MW = a^2 : b^2.$$

Again from (1)

$$\frac{MV}{MW} = \text{duplicate ratio of } \frac{MV}{MK};$$

$$\therefore\ \frac{a}{b} = \frac{MV}{MK} = \frac{MK}{MW} = \frac{MK - MV}{MW - MK} = \frac{VK}{KW} = \text{similarly } \frac{LV}{LW}.$$

Hence L and K are the points dividing VW externally and internally in the ratio $a : b$. Hence the construction,

Divide VW externally and internally at the points L, K, in the ratio $a : b$, and find Q, Q' so that $CQ^2 = CQ'^2 = CL \,.\, CK$.

Q, Q' are real or imaginary according as $CL \,.\, CK$ is positive or negative.

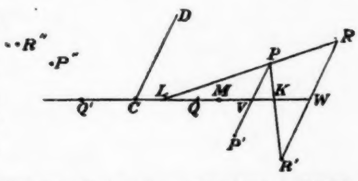
Secondly. If C be between V and W ($CV < CW$), make $CV' = CV$, and construct for $CV'W$; then since

$$V'Q \,.\, V'Q' = VQ \,.\, VQ',$$

the points Q, Q' satisfy the given conditions.

V. Theorem. *One only conic can be drawn having any two given parallel chords and its centre at any point on the line bisecting the chords; and the conic is a parabola when, and only when, the centre is at infinity.*

Let PP', RR' be the given chords, C the centre on VW bisecting them, CD parallel to PV. Then taking all possible cases in order,



(i.) Let PP', RR' be on the same side of CD; P, R in the same quadrant DCV, then PR must cut either CV or CD in this quadrant.

(a) Firstly. Let PR cut CV at L in the quadrant DCV; join PKR', and determine the real points Q, Q' such that $CQ^2 = CQ'^2 = CL \,.\, CK$, which is positive.

Now $$\frac{LV}{LW} = \frac{PV}{RW} = \frac{PV}{WR'} = \frac{VK}{KW}, \text{ by similar triangles};$$

$\therefore$ L, K are the points cutting VW externally and internally in the ratio $PV : RW$; and $CQ^2 = CQ'^2 = CL \,.\, CK$, by construction;

$$\therefore \text{ by IV., } \frac{VQ \,.\, VQ'}{WQ \,.\, WQ'} = \frac{PV^2}{RW^2};$$

$\therefore$ by II., the conic $\mu = PV^2 : VQ \,.\, VQ'$, and that only, passes through PP', RR', having C for centre.

(*b*) Secondly. Let PR cut CD and not CV in the quadrant DCV. Draw PP'', RR'' parallel to CVW to be bisected by CD.

Then the conic $PP''RR''$, centre C, passes through P', R', since CD, CV are conjugate.

(ii.) If RR' is not on the same side of CD as PP', draw RS, $R'S'$ parallel to CVW to be bisected by CD; then the conic $PP'SS'$, centre C, passes through RR', since CD, CV are conjugate.

(iii.) Since M, the mid-point of LK, uniquely satisfies the condition (see IV.), $MV : MW = PV^2 : RW^2$, the parabola $PV^2 = \mu \,.\, MV$, and that alone, passes through PP', RR'.

And if the diameter MV meet this again in M',

$$\frac{VM \,.\, VM'}{WM \,.\, WM'} = \frac{PV^2}{RW^2} = \frac{VM}{WM};$$

$$\therefore \frac{VM'}{WM'} = 1, \text{ and } M' \text{ is at infinity};$$

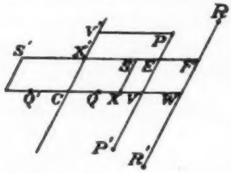
$\therefore$ the centre bisecting MM' is also at infinity.

With regard to degenerate conics, it will be readily seen that the above construction includes as special cases,

(i.) When the centre is at K or L, two right lines through centre;

(ii.) When PP' and RR' are equal, either (centre at mid-point of VW) any number of ellipses and hyperbolas, or (centre elsewhere) two parallel lines.

VI. THEOREM. *One only conic can be drawn through the extremities of any two parallel chords, to pass through a given point.*



Let PP', RR' be the given chords bisected by V, W, and S the given point. Draw SX, SEF parallel to PP', VW.

$$\text{Firstly, if } \frac{ES}{FS} = \frac{EP \,.\, EP'}{FR \,.\, FR'}, \qquad (1)$$

draw the parabola through P, P', R, R', cutting VW in Q; and let $PV^2 = \mu \,.\, QV$,

$$\therefore \frac{PV^2}{\mu \,.\, QV} = \frac{RW^2}{\mu \,.\, QW} = \frac{RW^2 - PV^2}{\mu \,.\, VW};$$

and (1) may be written

$$\frac{PV^2 - SX^2}{\mu \,.\, XV} = \frac{RW^2 - SX^2}{\mu \,.\, XW} = \frac{RW^2 - PV^2}{\mu \,.\, VW};$$

$$\therefore \frac{PV^2}{\mu \,.\, QV} = \frac{PV^2 - SX^2}{\mu \,.\, XV} = \frac{SX^2}{\mu \,.\, QX}.$$

Hence S is on this parabola.

Secondly, if (1) be not true, determine S' on SEF so that

$$\frac{ES\,.\,ES'}{FS\,.\,FS'}=\frac{EP\,.\,EP'}{FR\,.\,FR'}\text{ ....}(2)$$

This gives a single value for the ratio $ES':FS'$, and therefore a single position for S', which cannot be at infinity since then $ES':FS'=$ unity, and relation (1) is true.

Draw $X'C$ parallel to PP', bisecting SS' in X', and meeting VW in C, which cannot be at infinity. Draw PV' parallel to CV.

Draw the conic, centre C, through P, P', R, R', meeting CV, suppose, in real points Q, Q';

$$\therefore\ \frac{PV^2}{VQ\,.\,VQ'}=\frac{RW^2}{WQ\,.\,WQ'}=\frac{RW^2-PV^2}{CW^2-CV^2};$$

and (2) may be written

$$\frac{PV^2-SX^2}{CV^2-CX^2}=\frac{RW^2-SX^2}{CW^2-CX^2}=\frac{RW^2-PV^2}{CW^2-CV^2};$$

$$\therefore\ \frac{PV^2}{VQ\,.\,VQ'}=\frac{PV^2-SX^2}{CV^2-CX^2}=\frac{SX^2}{XQ\,.\,XQ'}.$$

Hence S is on this conic.

And if the conic do not meet CV in real points, it must meet its conjugate CX' in real points D, D'.

Then we can prove, as before,

$$\frac{PV'^2}{V'D.V'D'}=\frac{SX'^2}{X'D.X'D'}.$$

Hence S is on the conic.

VII. Theorem. *One only conic can be drawn through any five points.*

Let the points (see previous figure) be R, R', S, S', P; determine P' in the parallel through P to RR' so that

$$\frac{ES\,.\,ES'}{FS\,.\,FS'}=\frac{EP\,.\,EP'}{FR\,.\,FR'},\text{ ....}(1)$$

This gives a single value for EP', and therefore a single position for P'. It should be noticed, however, that in general the line bisecting PP', RR' will not be, as in the figure, parallel to SS'.

Then the conic through P, P', R, R', S meets ES in a point S_1, such that $\dfrac{ES\,.\,ES_1}{FS\,.\,FS_1}=\dfrac{EP\,.\,EP'}{FR\,.\,FR'}$;

$$\therefore\text{ by (1) }\frac{ES_1}{FS_1}=\frac{ES'}{FS'},\text{ and }S_1\text{ coincides with }S';$$

hence the conic passes through S'. Conversely, a conic through R, R', S, S', P must pass through P'. Hence there is only one conic.

E. Budden.

A CLASS OF ALGEBRAIC FUNCTIONS.

THE functions in question are those which involve only the differences of their arguments; and the object of this note is to suggest that an elementary discussion of these functions would be a valuable addition to the usual school course in algebra.

In his memoir on the Schwarzian derivative,* Cayley gives the name 'diaphoric' to these functions (Art. 13). Invariants, seminvariants, covariants of quantics are diaphoric functions of the roots. Norms of complex numbers and 'circulants' are diaphoric functions, a particular case of which already appears in books on algebra under the properties of cube roots of unity.

But quite apart from any applications, the diaphoric functions are remarkable for the facility with which they can be transformed, and variety of forms that can be obtained; and my suggestion is, that the theory is worthy of a place beside that of symmetrical functions.

It might be treated somewhat in this way:

I. A diaphoric function of $x, y, \dots z$ is unchanged when any quantity δ (the same for all the arguments) is subtracted from the arguments $x, y, \dots z$.

For every difference, $x-y$ is unchanged.

II. If a function of $x, y, \dots z$ is unchanged when any quantity δ is subtracted from each of the arguments, whatever value δ may have, it is diaphoric.

For we may give to δ the value of one of the arguments, and then the function is visibly diaphoric.

III. This gives a simple mode of deciding whether a given function is or is not diaphoric. As an example, take

$$U=x^2+2y^2+3z^2-4yz-2zx.$$

If this is diaphoric in x, y, z, it is unchanged when z is subtracted from each argument, giving

$$U'=(x-z)^2+2(y-z)^2,$$

which on expansion is found to be the same as U.

Hence $$U=(x-z)^2+2(y-z)^2,$$

and is visibly diaphoric.

But if U' is different from U, the latter cannot be diaphoric, seeing that for a value of δ it does not satisfy the test.

It will be observed that one trial suffices to decide the point.

IV. When U is known to be diaphoric, expressions equal to U but of different forms can at once be written down by assuming different values of δ, and by linear combinations of these

* *Collected Mathematical Papers*, vol. xi., p. 156.

expressions. For example, taking U as above, we have

$$U=3(y-z)^2+(x-y)^2-2(z-y)(x-y) \qquad (\delta=y)$$
$$=2(x-y)^2+3(z-x)^2-4(y-x)(z-x) \qquad (\delta=x)$$
$$=4x^2+2(x+y)^2+3(z+x)^2-4(x+y)(x+z)-4x(x+z) \qquad (\delta=-x)$$
$$=(x+y-z)(x-y-z)+2x^2+3y^2-4yz, \text{ etc., etc.}$$

V. We sometimes simplify the analysis of a given expression by assigning special values to one or more of the arguments. In the case of diaphoric functions there is the important advantage that, if we assign the value 0 to any argument, say x, the original function is immediately found again by replacing the remaining arguments $(y, \dots z)$ by $y-x, \dots z-x$.

It is obvious that when a diaphoric function is not homogeneous in the arguments it can always be expressed as a sum of diaphorics each of which is homogeneous, and there is no loss of generality in restricting ourselves to homogeneous diaphorics.

VI. Integral algebraical diaphoric functions have the property that every integral factor involving any of the arguments is also diaphoric.

Suppose W is a diaphoric function of $x, y, \dots z$, and is a product of two factors U, V. We shall show that U, V are diaphoric functions. When δ is subtracted from each argument, U, V become

$$U+U_1\delta+U_2\delta^2+\dots+U_r\delta^r, \quad V+V_1\delta+\dots+V_s\delta^s$$

if r, s denote the degrees of U, V respectively. Also W becomes W',

$$W'=(U+U_1\delta+\dots+U_r\delta^r)(V+V_1\delta+V_2\delta^2+\dots+V_s\delta^s).$$

But $W'=W=UV$, so that

$$0=(UV_1+U_1V)\delta+\dots+(UV_t+U_1V_{t-1}+\dots+U_tV)\delta^t$$
$$+\dots+U_rV_s\delta^{r+s}$$

for all values of δ. Hence the coefficient of each power of δ vanishes; and this implies, as we shall show, that $U_1, U_2 \dots U_r$, $V_1, V_2, \dots V_s$ all vanish.

To fix the ideas, suppose $r=3, s=4$. We have then the following equations to satisfy:

$$U_3V_4=0, \qquad (7)$$
$$U_3V_3+U_2V_4=0, \qquad (6)$$
$$U_3V_2+U_2V_3+U_1V_4=0, \qquad (5)$$
$$U_3V_1+U_2V_2+U_1V_3+UV_4=0, \qquad (4)$$
$$U_3V+U_2V_1+U_1V_2+UV_3=0. \qquad (3)$$

From (7) it appears that U_3 or V_4 is 0, and from the other equations it may be inferred that both vanish. For if $V_4 \neq 0$, $U_3=0$, and therefore (6) gives $U_2V_4=0$, *i.e.* $U_2=0$. In like manner (5) gives $U_1=0$, and (4) $U=0$, which is absurd. Thus

$V_4=0$. In the same way, the supposition that $U_3 \neq 0$ leads to $V=0$, which is impossible, and therefore $U_3=0$.

Having $U_3=0$, $V_4=0$, we may omit all terms containing these letters; and we obtain with the equation

$$U_2V+U_1V_1+UV_2=0, \qquad (2)$$

a system of four equations similar to the original system which gives $U_2=V_3=0$.

Again, omitting the vanishing terms, and using the equation

$$U_1V+UV_1=0, \qquad (1)$$

we find $U_1=V_2=0$, and then (1) yields $V_1=0$.

Now, in this analysis U is *any* factor (involving some of the arguments of W), and V is the co-factor. Hence we have shown that any factor U of a diaphoric function (W) is diaphoric. It will be observed that the proof though applied to a particular case is really quite general.

[It is perhaps proper to add that the theorem cannot be proved by means of the differential equation satisfied by diaphoric functions in general, and, indeed, is not true for such functions. Suppose, for example, U and V are diaphoric functions, so that e^{U+V} is likewise diaphoric. Then we have

$$e^{U+V}=e^{U+W}\times e^{V-W},$$

where W is any quantity whatever, and need not be diaphoric.]

VII. A few examples follow:

EXAMPLE 1. Factorize $\Sigma(b-c)^3=U$.

$$\begin{aligned} U' &= b^3-a^3+(a-b)^3 \quad \text{(when } c=0) \\ &= 3ab^2-3a^2b \quad \text{"} \\ &= 3ab(b-a); \quad \text{"} \end{aligned}$$

$$\therefore\ U=3(a-c)(b-c)(b-a)=3(b-c)(c-a)(a-b).$$

EXAMPLE 2.

$$U=\Sigma(b-c)^5;$$

$$\begin{aligned} \therefore\ U' &= b^5-a^5+(a-b)^5 \quad (c=0) \\ &= 5ab(-a^3+2a^2b-2ab^2+b^3) \quad \text{"} \\ &= 5ab(b-a)(b^2-ab+a^2); \quad \text{"} \end{aligned}$$

$$\begin{aligned} \therefore\ U &= 5(a-c)(b-c)(b-a)\{\Sigma a^2-\Sigma bc\} \\ &= 5(b-c)(c-a)(a-b)\{(c-a)^2+(b-c)^2+(c-a)(b-c)\}. \end{aligned}$$

EXAMPLE 3. The expression $\Sigma a^2-\Sigma bc$, obtained in Example 2, being diaphoric in three arguments, is really a quadratic function of two arguments; and is therefore factorizable. The factors must be of the form $(a+hb+kc)(a+kb+hc)$, since the expression is symmetrical in (b, c). Also since each factor is diaphoric, $1+h+k=0$, and observing the coefficient of b^2 (or c^2) in the product $hk=1$;

$$\therefore\ 1+h+h^{-1}=0,$$

and the factors are $(a+hb+h^{-1}c)(a+h^{-1}b+hc)$.

EXAMPLE 4. Determine A, B, C so that

$$U=Axyz+B(x+y+z)(yz+zx+xy)+C(x+y+z)^3$$

may be diaphoric.

Here $U' = B(x+y)xy + C(x+y)^3 \quad (z=0)$;

and, if diaphoric, $U = B(x+y-2z)(x-z)(y-z) + C(x+y-2z)^3$.

Hence U is a multiple of $x+y-2z$, and, by symmetry,

$$U = k\Pi(x+y-2z);$$

$$\therefore\ U' = B(x+y)xy + C(x+y)^3 = k(x+y)(y-2x)(x-2y),$$

$$Bxy + C(x+y)^2 = k(-2x^2+5xy-2y^2);$$

$$\therefore\ C = -2k,\quad B+2C = 5k,\quad B = 9k.$$

Again, putting $x=y=z=1$, so that $U=0$,

$$0 = A + 9B + 27C;$$

$$\therefore\ A = -81k + 54k = 27k;$$

$$\therefore\ U, = (-27xyz + 9\Sigma x\,.\,\Sigma yz - 2(\Sigma x)^3)k, = k\Pi(x+y-2z),$$

if it is diaphoric.

Hence if x, y, z are roots of an equation $ax^3+3bx^2+3cx+d=0$,

then $$U = 27\frac{k}{a^3}(a^2d - 3abc + 2b^3) = 27k\left(-\frac{b}{a}-x\right)\left(-\frac{b}{a}-y\right)\left(-\frac{b}{a}-z\right),$$

$$\frac{a^3}{27k}U = a^2d - 3abc + 2b^3 = -(ax+b)(ay+b)(az+b).$$

H. W. LLOYD TANNER.

CAYLEY'S THEORY OF THE ABSOLUTE.

THIS is the title of an extremely interesting paper by PROFESSOR CHARLOTTE ANGAS SCOTT, in the *Bulletin* of the American Mathematical Society, 2nd Series, Vol. III., No. 7. Its object is "to show, as a matter of purely pedagogic interest, how simply and naturally Cayley's theory of the Absolute follows from a small number of very elementary geometrical conceptions, without any appeal to analytical geometry." The "contention is not that every step in the rigorous proof can be presented under the guise of elementary mathematics, but that it is quite possible to develop the theory so as to be intelligible and interesting to average students at a much earlier stage than is customary." The paper begins by explaining the principles of projection. If a plane figure be projected in any manner into another plane figure, it is easily seen "that the properties in which any difference is to be observed in the two figures all depend on measurement of lines or angles; these properties are called metric; the properties that are unaffected by the change from one figure to the other, unaffected, that is, by projection, are called projective." The simplest projective properties are that points project into points and straight lines into straight lines, and consequently collinear points into collinear points and concurrent lines into concurrent lines. By *assuming* that these general laws admit of no exception, and examining by what conception or artifice *apparent* exceptions may be brought under the general rule, we are led to postulate the existence in any

plane of the straight line at infinity (all points at infinity in a plane being the projections of points on a straight line in another plane), and to the result that parallel lines form a concurrent system, the point of concurrence being at infinity.

We have noticed the distinction between metric and projective properties. Metric geometry has two main objects: (i.) to distinguish the position of points and lines, and to refer to them by means of numbers, and (ii.) to compare magnitudes of the same kind in different positions. Can these objects be attained in purely projective geometry? The answer is, that (i.) can be by means of the conception of the *Absolute*; but certainly not (ii.), except in a very modified sense, since projective geometry has nothing whatever to do with absolute magnitudes. Thus the Absolute is merely postulated in order that the positions of points and lines may be referred to projectively. The Absolute cannot be said to have any actual existence, and it does not throw any light on the nature of space; the most that can be expected from it is a simplification of theoretical geometry.

To examine how the relative positions of points may be determined projectively, *i.e.* without any considerations of length, we start with points on a straight line. Let A, B, C be any three given points on a straight line. They certainly have one property that is projective, viz. the fact that they are collinear, but they have no other projective property. This is proved by showing that two projections are sufficient to change A, B, C into any other set of three points A', B', C' on a line. Project A, B, C from any point on CC' on to the line AC', giving A, X, C'; then project A, X, C' into A', B', C', viz. from the point where AA', XB' meet. This construction holds whether ABC, $A'B'C'$ are in the same plane or not. The result is that if A, B are two given points determining a straight line, there is no projective property for a third point C on AB, which will determine the position of C. The case is, however, different with four points. If A, B are two given points, there is a projective property (as proved below) for any other pair of fixed points P, Q on AB. This property may be expressed by saying that to every pair of fixed points P, Q on AB there is, in reference to the two given points A, B, a corresponding number $\overline{PQ}$, which can be chosen in such a way that the relation $\overline{PQ}+\overline{QR}=\overline{PR}$ shall always be true. The number $\overline{PQ}$ may be called the generalized or projective distance from P to Q.

"Similarly we can (projectively) measure angles about a point if we have a pair of lines a, b through the point as a standard of reference. The generalized angular distance from any line p to any line q, that is, the generalized measure of the angle made by q with p, is written $\overline{pq}$. And just as in the case of the linear

measurements, these angles are subject to the law expressed by the equation $\overline{pq}+\overline{qr}=\overline{pr}$."

"Thus we can construct a (projective) system of measurement on a line if we have on that line a certain absolute configuration, two fixed points; and a system of measurement about any point if we have an absolute configuration, two fixed lines through the point. In attempting to apply this to a plane, we require an absolute configuration that shall give us two points on every line, and two lines through every point."

This absolute configuration must clearly be a curve which every straight line cuts in two points, real or imaginary, and to which two tangents can be drawn from every point; *i.e.* it must be a conic. The Absolute in a plane is therefore a fixed conic, and provided it be given it becomes possible to classify the points and lines of the plane projectively, and to indicate their relative positions by numbers. The paper goes on to show that in the geometry of Euclid the Absolute must be a degenerate conic consisting of a pair of points, viz. the circular points at infinity.

We have deferred to the end the consideration of the fundamental theorem that four points on a straight line have a relation to one another which is unaltered by projection. In order that projective geometry may be complete in itself this theorem must of course be proved by projective methods, without any reference to length. Now there is a certain standard configuration of four points on a straight line such that if three of them be chosen at will the fourth can be uniquely determined. This arrangement of four points consists of two pairs, viz. the ends A, B of any diagonal of a complete quadrilateral, and the points C, D in which the two other diagonals cut AB. The points A, B, C, D are said to form a harmonic range. If three points A, B, C be given, an infinite number of quadrilaterals can be described having AB for a diagonal, and such that a second diagonal passes through C, but the third diagonal then always cuts AB in the same fixed point D. This important theorem, which is essential for the theory developed in the paper, seems to be overlooked. The proof depends on the well-known property in projective geometry that two triangles, in the same plane or not, which have an axis of perspective, have also a centre of perspective, and conversely (see Cremona's *Projective Geometry*, Arts. 16, 17, 36 (5), 46). Now a harmonic range projects into a harmonic range, as is seen at once by projecting the complete quadrilateral by which the range is constructed; hence a harmonic range is a standard configuration in projective geometry. Further, if A, B, C be any three given points on a line, we can obtain an infinity of other points on the line by a series of harmonic constructions, and we *assume* that we can thus exhaust all the points on the line. In other words we assume that any

fourth point K on the line is connected with A, B, C by a finite or infinite number of harmonic constructions. It follows that if A', B', C' be any other given set of three points on a line there is a fourth *unique*, real or ideal, point K' such that A', B', C', K' is projective with A, B, C, K. Thus, any four given points A, B, C, K on a line have a relation to one another, other than that of collinearity, which is unaltered by projection.

This theorem enables us to give a definition of *equal* projective distances, whether on the same line or on different lines. Two distance $\overline{PQ}$, $\overline{P'Q'}$ are said to be equal if P, Q, Y, Z, and P', Q', Y', Z' are projective with one another, Y, Z and Y', Z' being the points in which the lines PQ and $P'Q'$ cut the Absolute. Thence we easily pass to the comparison of any commensurable distances. We choose any fixed distance $\overline{AB}$ for unit, and suppose it divided into any integral number (n) of equal parts AA_1, A_1A_2, ... $A_{n-1}B$. Then any distance $\overline{PQ}$ which is made up of m parts equal to AA_1 is measured by the number m/n; and the ratio of any two distances is defined to be the ratio of the numbers which correspond to them. Further, the addition law $\overline{PQ}+\overline{QR}=\overline{PR}$ clearly holds. The extension to incommensurable distances follows in the usual way.

On account of its admirable clearness and suggestiveness the paper will well repay perusal, although a more complete reference to the above theorem and its consequences might advantageously have been given.

F. S. M.

MATHEMATICAL NOTES.

49. *Notes connected with the Analytical Geometry of the Straight Line.*

I have found that the notion of the "gradient"* of a line is of great assistance to a beginner. He readily realizes how it is measured in terms of the coordinates of two points on the line, viz. by the ratio of the difference of ordinates to the difference of abscissae (when the axes are rectangular), and he appreciates the distinction between a positive and a negative gradient. The term is itself a good one, as the one word *gradient* does duty for the expression *tangent of slope*, or *tangent of the angle made with the axis of x.* It is necessary to assume, for the complete realization of the notion, that the axis of x is horizontal—but as this is how it is generally drawn in text-books the assumption is not serious. Indeed, it finally is hardly necessary, for in dealing with the tangent of the angle between two lines it may be called the *relative* gradient of one with regard to the other, which is perfectly understandable, and, moreover, enables us to attach a definite meaning to the + or − sign which appears in the numerical values of this relative gradient.

This fundamental idea of gradient, *i.e.* of x-gradient, or gradient with regard to the axis of x, is of great importance also when the student goes on to tangents to curves. Moreover, here the notation shows to advantage. Instead of having to speak of *the tangent of the angle which the curve or the tangent to the curve makes with the axis of x*, we may simply say *the gradient*

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* Used in Prof. Perry's *Calculus*.

of the curve or tangent. The use of the word 'tangent' in different senses in thus avoided.

In oblique coordinates also the expression for the gradient is most important, and is easily obtained straight from a figure. It is

$$\frac{(y-y')\sin\omega}{(x-x')+(y-y')\cos\omega},$$

the rectangular expression being $\frac{y-y'}{x-x'}$, where (x, y) and (x', y') are the coordinates of two points on the line.

(If we wish to write the cotangent of the angle of slope instead of the tangent, the word 'co-gradient' might be found useful. Thus, if two lines are at right angles, the gradient of one is equal to minus the co-gradient of the other.)

Armed with this expression and with the expression for the distance between two points, there is but little book work, and there are but few of the early problems, which we cannot at once tackle with comparative ease and certainty.

Of course there are other expressions and methods of working which come later; all I wish to emphasize is the precision and power which this simple weapon confers, and its importance in later (even calculus) work as well as at the beginning.

I would advocate, so as to keep the gradient idea prominent, that the equation of a line through two points should be written

$$\frac{y-y'}{x-x'}=\frac{y''-y'}{x''-x'}$$

instead of in the way usual in text-books.

To find the gradient of a line whose equation is given, *e.g.* $2x-3y+7=0$, we have, if (x', y') is any point on the line, $2x'-3y'+7=0$;

$$\therefore\ 2(x-x')-3(y-y')=0; \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}$$

$$\therefore\ \frac{y-y'}{x-x'}=\frac{2}{3}=\text{gradient, if the axes are rectangular.}$$

If the axes are oblique, the gradient

$$=\frac{(y-y')\sin\omega}{(x-x')+(y-y')\cos\omega}=\frac{2(y-y')\sin\omega}{3(y-y')+2(y-y')\cos\omega}, \text{ from (i.)}, =\frac{2\sin\omega}{3+2\cos\omega}.$$

This example is given to illustrate the use of the gradient expressions.

With regard to the important expression in rectangular coordinates,

$$P=\frac{ax'+by'+c}{\sqrt{a^2+b^2}},$$

I would like to suggest the following proof, as the expressions used in the proof are themselves of some importance and interest:

Let X, Y denote respectively the horizontal and vertical distances of (x', y') from $ax+by+c=0$.

Then $$X=x'-x'', \text{ where } ax''+by'+c=0,$$

$$\therefore\ X=\frac{1}{a}(ax'+by'+c).$$

Similarly $$Y=\frac{1}{b}(ax'+by'+c).$$

Also $$\left(\frac{P}{X}\right)^2+\left(\frac{P}{Y}\right)^2=1, \text{ by elementary trigonometry,}$$

$$\therefore\ (aP)^2+(bP)^2=(ax'+by'+c)^2,$$

which gives us P at once.

50. *On the cumulative vote as exercised in School Board elections.*

The number of ways in which an elector can distribute r votes among n candidates is the coefficient of x^r in $(1-x)^{-n}$. Hence the total number of ways in which an elector can vote when k seats are to be filled

$$=\text{sum of coefficients of } x, x^2, x^3, \dots x^k \text{ in } (1-x)^{-n}$$

$$=\text{coefficient of } x^k \text{ in } (1-x)^{-n-1}-1$$

$$=\frac{(n+1)(n+2)\dots(n+k)}{\underline{|k}}-1$$

$$=(\text{number of selections of } k \text{ things out of } n+k)-1.$$

The following investigation of the *à priori* grounds for the form of the result may be of interest:

Let (k, r) denote the number of ways in which an elector, having selected r candidates, may distribute his k votes among them, giving each at least one. Taking the case of any particular candidate of these r, he must receive 1, 2, 3, or $k-(r-1)$ votes; and the number of ways the remaining $k-1$, $k-2$, $k-3$, ... or $r-1$ votes can be given is $(k-1, r-1)$, $(k-2, r-1)$, $(k-3, r-1)$, ... $(r-1, r-1)$ respectively.

Hence $(k, r)=(k-1, r-1)+(k-2, r-1)+(k-3, r-1)\dots+(r-1, r-1)$.

But this equation would hold if (k, r) denoted the number of selections that could be made out of k things a, b, c, etc., r at a time, for this must be made up of (all that we could make containing a)+(all that we could make containing b and not a)+(all that we could make containing c but not a and b), and so on.

Also $(k, 1)=k=$number of ways of selecting one thing out of k.

Hence $(k, r)=$number of selections out of k things r at a time.

Now, if n be the whole number of candidates for k seats, each voter can *select* r candidates in (n, r) ways, and may vote for them in (k, r) ways after he has selected them.

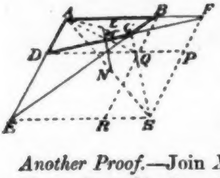
Hence whole possible number of ways of voting

$$=(n, 1).(k, 1)+(n, 2).(k, 2)+(n, 3).(k, 3)+\dots+(n, k).(k, k)$$
$$=(n, 1).(k, k-1)+(n, 2).(k, k-2)+(n, 3).(k, k-3)+\dots+(n, k).1.$$

Now, any term $(n, r).(k, k-r)$ represents the number of ways of selecting r candidates out of a group n and $k-r$ candidates out of a group k, and therefore the whole expression gives the number of possible ways of selecting k candidates out of $n+k$, except that we are to take at least one from group n [the term $1.(k, k)$ not appearing].

Hence whole possible number of ways$=(n+k, k)-1$. E. M. Langley.

51. *The mid-points of the three diagonals of a complete quadrilateral are collinear.*



Through D, E draw DQP, ERS parallel to AB, and through F, B draw FPS, BQR parallel to AD.

Then it is known that the diagonals BE, DF, QS co-intersect, *i.e.* that S, Q, C are collinear.

Hence the mid-points L, M, N of AC, AQ, AS are on a line parallel to CS. But M, N are also the mid-points of BD, EF. E. M. Langley.

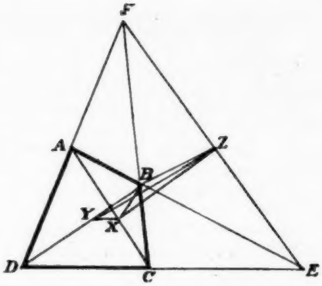
Another Proof.—Join XY, XZ, and YZ.

Then, since Z is the mid-point of EF,

$$2BYZ=EBY-FBY$$
$$=\tfrac{1}{2}EBD-\tfrac{1}{2}FBD. \qquad \text{(1)}$$

Similarly
$2BXY = CBY - ABY$
$= \frac{1}{2}CBD - \frac{1}{2}ABD$; ....(2)
and also
$2BXZ = EBX - FBX$
$= \frac{1}{2}BCE - \frac{1}{2}FAB$. ....(3)
But right-hand side of (1) = sum of right-hand sides of (2) and (3);
$\therefore$ $BYZ = BXY + BXZ$;
$\therefore$ XYZ is of zero area;
$\therefore$ X, Y, Z are collinear.

A. E. DANIELS.



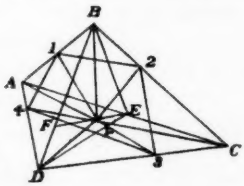
52. *The locus of the centres of conics inscribed in a given quadrilateral is the straight line joining the mid-points of the diagonals.*

Take a conic, centre P, touching the sides of the given quadrilateral $ABCD$ in the points 1, 2, 3, 4.

Then PA bisects 14, and therefore

$\triangle P1A = \triangle P4A$.
Similarly $\triangle P1B = \triangle P2B$,
$\triangle P2C = \triangle P3C$,
$\triangle P3D = \triangle P4D$;

$\therefore$ difference between $\triangle PAB$ and $\triangle PBC$
= difference between $\triangle PAD$ and $\triangle PDC$.



Again, take E, the mid-point of AC.

Difference between $\triangle PAB$ and $\triangle PBC = 2\triangle PBE$,
and difference between $\triangle PAD$ and $\triangle PDC = 2\triangle PDE$;
$\therefore$ $\triangle PBE = \triangle PDE$;
$\therefore$ EP produced bisects BD (in F say);
$\therefore$ P lies in line joining mid-points of diagonals of the given quadrilateral.

A. E. DANIELS.

53. *The in-centre is the point for which Σap^2 is a minimum in each of two cases:*

(1) *if p, q, r be the perpendiculars from a point O in the plane of a triangle on its sides;*

(2) *if p, q, r are the joins of O to the vertices.*

(1) $$\Sigma ap^2 = \Sigma ap^2(a+b+c)/2s$$
$$= \{\Sigma a^2p^2 + \Sigma p^2(ab+ac)\}/2s$$
$$= \{(\Sigma ap)^2 + \Sigma ab(p-q)^2\}/2s;$$

but Σap is a constant, viz. twice the area of the triangle;

$\therefore$ Σap^2 is a minimum for $p=q=r$, *i.e.* for in-centre.

(2) If x, y, z are the lengths of the perpendiculars on the sides, we have
$$p^2\sin^2 A = y^2 + z^2 + 2yz\cos A;$$
$\therefore$ Σap^2 is a minimum when $\Sigma \frac{1}{a}(y^2+z^2+2yz\cos A)$ is a minimum,

i.e. when $\frac{1}{abc}(\Sigma ax)^2 + \Sigma \frac{1-\cos A}{a}(y-z)^2$ is a minimum, *i.e.* when $x=y=z$;

$\therefore$ Σap^2 is a minimum for the in-centre.

W. J. GREENSTREET.

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the GAZETTE *attractive by sending either original or selected problems.*

Solutions should be sent within three months of the date of publication. They should be written clearly on one side of the paper. Contractions not intended for printing should be avoided. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

The question need not be re-written, but the number should precede every solution.

The source of problems when not otherwise indicated is shown by —C. (Cambridge), O. (Oxford), D. (Dublin), W. (Woolwich), Sc. (Science and Art Department).

221. The opposite sides of a hexagon are parallel; their extremities are joined and intersect in three points, the joins of which are parallel to the sides of the hexagon. J. BLAIKIE.

222. Prove that

$$(2n)^3(r-2)mP+(r-4)^2(4n^3-n)/3$$
$$=\sum_{K=1}^{K=2n-1}\{2mn(r-2)-K(r-4)\}^2,$$

where P is any polygonal number, m its root, r its order,

e.g. $r=3, n=1, \quad 8mT+1=(2m+1)^2$

$r=5, n=3, 648mP_5+35=(18m-1)^2+(18m-3)^2+(18m-5)^2.$

R. W. D. CHRISTIE.

223. Pairs of orthogonal circles pass through two fixed points; show that their common tangents envelop a fixed ellipse, and that the points of contact lie on two fixed lines. E. FENWICK.

224. APB, AQC are circles touching AC, AB respectively. AR any chord of the circumcircle of ABC cuts these circles in P and Q. Show that $AR=AP+AQ$. R. W. GENESE.

225. P, Q are points of inverse coordinates $\left(i.e.\ h,\ k,\ \frac{a^2}{h},\ \frac{b^2}{k}\right)$ with respect to the axes of an ellipse. Tangents are drawn to the ellipse from either, and the normals at the points of contact meet in O.

Prove (a) that CO is perpendicular to PQ, and that $\frac{CO}{PQ}$ is the cotangent of the angle between CP and any chord which it bisects; (b) the director-circle and the circle on PQ as diameter are orthogonal. Hence show how to draw 4 normals from any given point in a diameter at right-angles to either of the equi-conjugates. E. P. ROUSE.

226. ABC is an acute-angled triangle, P its orthocentre. The product of the latera-recta of three circumscribing ellipses, having

the sides respectively as minor axes is $AP.BP.CP$. Discuss the corresponding case of ellipses and hyperbolas when triangle is obtuse. G. HEPPEL.

227. EF is the third diagonal of $ABCD$. H, K, L, M are points on AB, DC, BC, AD such that $EH=AB$, $EK=CD$, $FL=BC$, $FM=DA$. Then $HKLM$ is a parallelogram.

E. M. LANGLEY, Jun.

228. OP touches the circle APB, OAB is a secant through the centre. OL, OM are taken along OP equal to OA, OB respectively. PQ is perpendicular to AB; LX, MY to OAB. Show $LX=AQ$, $MY=BQ$. E. M. LANGLEY.

229. OP is perpendicular to the plane of the rectangle $APBC$. The solid angle ω subtended by $APBC$ at O is given by $\sin\omega=\sin\alpha\sin\beta$, where α, β are the angles subtended at O by PA, PB. A. LODGE.

230. Two points P, Q, on the sides of any triangle are opposite corners of a rhombus, the others being the incentre and the vertex. Show that the circle touching the sides at P and Q touches the circle through the ends of the base and the circumcentre. C. E. M'VICKER.

231. Two concentric and co-axial ellipses are so related that triangles circumscribed to the one are inscribed to the other. The normals to the former at the points of contact of the sides of any one of the triangles are concurrent, as also are the normals to the other ellipse at the angular points of the triangle.

G. RICHARDSON.

232. The polars of A, B, C with respect to semicircles on BC, CA, AB cut AB, AC in Q, P; BC, BA in R, S; CA, CB in V, W.

(1) The polars meet in the orthocentre H.
(2) BP, CQ meet in K on the perpendicular AD, and so on.
(3) E, K, W; F, K, R; etc., are collinear.
(4) LD, MD are equally inclined to AD, and so on.

R. TUCKER.

233. A, B, and C each arrange 105 square tiles (side $=1$ decimetre) so that the outer and inner boundaries of the patterns are squares. Now the three patterns are all different and the outer boundary of A's pattern is as much longer than the inner boundary of B's pattern as the outer boundary of B's pattern is shorter than the inner boundary of C's pattern. Give the dimensions of A's pattern. H. W. LLOYD TANNER.

234. Three fine light rods, AB, AC, AD, are freely jointed at A, and rest in a vertical plane on smooth horizontal supports at B and C under a load applied at A. The rod AD rests vertically downwards, and its extremity is connected with B and

C by two fine light strings BD, CD. Prove that if T is the tension of the rod AD, and W the weight of the load at A, then $T : W = OD : DA$ where O is the intersection of BC and AD produced. W. J. DOBBS.

235. If $\sum_{r=1}^{n-1} x^2_r - \sum_{r=1}^{n-2} x_r x_{r+1} = c$, prove that x^2_r is not greater than $2cr\left(1-\frac{r}{n}\right)$; and that if $r<n-r$, the limits of $x_r x_{n-r}$ are cr and $-cr\left(1-\frac{2r}{n}\right)$. F. S. MACAULAY.

236. If the smallest prime factor of a number not greater than m^n be not less than m, then the number contains not more than $n-1$ prime factors. E. HILL.

237. Show that if an ellipse, parabola, and hyperbola have the same directrix and vertex, the ellipse will be entirely within the parabola, and the parabola entirely within the hyperbola.
T. ROACH.

238. The area of any plane triangle whose sides are integers is divisible by 6. ARTEMAS MARTIN.

239. In the process of converting 1/100103 into a repeating decimal the 273rd remainder is 100067, the 5005th is 23447, the 6126th is 14, the 6290th is 26789, and the 7587th is 100091. Find where the remainders 1, 2, 3, ... 10 will occur. (A correction of 149.) C. N. MURTON.

240. If $\alpha, \beta, \gamma \ldots$ be random magnitudes subject to the condition that their sum is S, show that

$$\epsilon(\alpha^p \beta^q \gamma^r \ldots) = \frac{\underline{|n-1} \; \underline{|p} \; \underline{|q} \; \underline{|r} \ldots}{\underline{|n-1+p+q+r+\ldots}} \cdot S^{p+q+r+\ldots}$$

where $p, q, r \ldots$ are any integers and $\epsilon(x)$ denotes the expectation of x. W. ALLEN WHITWORTH.

SOLUTIONS.

A great number of solutions are in hand, and will be published whenever sufficient space is available. Solutions are wanted for the following: Nos. 90, 131, 144, 150, 170, 171, 172, 194, 219 (see correction below to 219).

ERRATA.

Question 219, p. 133: write $\frac{c}{a}$ for $\frac{c}{2}$.
p. 137, line 19 up: after 'H' insert 'which are.'
" " 18 up: for 'HP, HQ' read 'HQ, HP.'
" " 11 up: after 'on' insert 'HP and similarly on.'

197. *If $ABCD$ is a skew quadrilateral (that is, a quadrilateral whose sides are not in one plane) with the angles A, B equal, and also the angles C, D equal, then $AD=BC$ and $AC=BD$.* R. F. MUIRHEAD.

Solution by Prof. A. C. DIXON and C. E. M'VICKER.

On BC make $BE=AD$. Then $AE=BD$ and $\angle ADE=\angle BED$. Thus

$$\angle ADE=\angle BCD+\angle CDE=\angle ADC+\angle CDE,$$

so that DA, DE, DC are co-planar and C, E coincident.

198. *If, in the quadrilateral of the preceding problem, E is the middle point of AB, and F that of CD, then EF is perpendicular both to AB and to CD.* R. F. MUIRHEAD.

Solution by Prof. A. C. DIXON and C. E. M'VICKER.

The generating parallelopiped of the tetrahedron $ABCD$ has two pairs of rectangular faces, and E, F are the centres of the other faces. Hence EF is perpendicular to these other faces, *i.e.* both to AB and CD.

199. *A has a counters and B has b counters; what is A's chance of winning all of B's counters, the play being even?* W. H. BESANT.

Solution by the PROPOSER.

When A has n counters, let u_n be his chance of success. Then, as A either wins or loses the next game,

$$u_n=\tfrac{1}{2}u_{n+1}+\tfrac{1}{2}u_{n-1} \quad\text{or}\quad u_{n+1}-2u_n+u_{n-1}=0.$$

Hence, observing that $u_0=0$, we have

$$u_n-u_{n-1}=u_{n-1}-u_{n-2}=u_{n-2}-u_{n-3}=\ldots=u_2-u_1=u_1;$$

$$\therefore\quad u_n=n\,.\,u_1,\ \text{and since } u_{a+b}=1,\ u_1=\frac{1}{a+b};$$

$$\therefore\quad u_n=\frac{n}{a+b},\ u_a=\frac{a}{a+b}.$$

200. *A number (n) of men, accompanied by a number (r) of tame monkeys, pick bananas, pile up the heap, and go to sleep. During the night one man gets up, gives each monkey a banana, divides the remainder of the heap into n equal portions, carries off and hides away one portion, piles up the heap again, and goes to sleep. A second man then wakes up, gives each monkey a banana, and, just as the first man did, takes away an n^{th} part of the remainder and goes to sleep. The rest of the men follow, do the same thing, and go to sleep. Finally, in the morning, all the men wake up, give each monkey a banana, and then find that the heap which is left can be divided into n equal portions. Find an expression for the least possible number of bananas; and prove that, for four men and one monkey,* 1021 *is the least possible number.* W. H. BESANT.

Solution by the PROPOSER, E. P. BARRETT, and R. F. DAVIS.

Let u be the original number of bananas in the heap which is diminished successively to u_1, u_2, ... u_n after each operation as described is performed. Then writing λ for $(n-1)/n$ we have

$$\begin{aligned}
&u_1=\lambda(u-r),\\
&u_2=\lambda(u_1-r), \qquad \text{also } u_n-r=M(n).\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&u_n=\lambda(u_{n-1}-r);
\end{aligned}$$

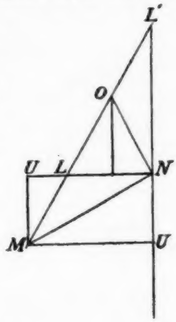
$$\begin{aligned}
\text{Thus}\qquad &\lambda^n u-r(\lambda^n+\lambda^{n-1}+\ldots+\lambda+1)=M(n),\\
\text{or}\qquad &\lambda^n u-nr(1-\lambda^{n+1}) \qquad =\ldots\ldots\\
&\lambda^n(u+\lambda nr) \qquad =\ldots\ldots\\
&(n-1)^n\{u+(n-1)r\} \qquad =n^n M(n).
\end{aligned}$$

Now $n-1$ and n being prime to each other, the least value of u occurs when $M(n)=n.(n-1)^n$, in which case $u=n^{n+1}-(n-1)r$. When $n=4$, $r=1$, then $u=1021$.

201. *The product of the latera recta of two parabolas through four concyclic points is* $\frac{1}{8}(d_1^2-d_2^2)\sin\omega$, *where* d_1, d_2 *are the diagonals of the quadrilateral formed by the four points, and* ω *the angle between them.* Jesus (C.), 1890.

Solution by R. F. Davis.

Let O be the centre of the circle through the four points; M, N the mid-points of the diagonals.



Then (*Parabola*, Milne and Davis, p. 38) the axes of the two parabolas are parallel respectively to the in- and ex-bisectors of $\angle MON$, and intersect at mid-point of MN; also MO and the ordinate of M intercept on the diameter (of the parabola) through N, a length equal to the latus rectum.

Hence in the figure UL, $U'L'$ are the two latera recta, and

$$UL=\tfrac{1}{2}(OM \backsim ON)\sin\tfrac{1}{2}MON;$$
$$U'L'=\tfrac{1}{2}(OM+ON)\,.\cos\tfrac{1}{2}MON.$$

But
$$MON=\pi-\omega;$$
$$OM^2=R^2-\tfrac{1}{4}d_1^2;$$
$$ON^2=R^2-\tfrac{1}{4}d_2^2;$$
$$\therefore\ UL\,.\,U'L'=\tfrac{1}{8}(d_1^2-d_2^2)\sin\omega.$$

Mr. Heppel sends an analytical solution.

$MNPQ$ the quadrilateral, PM, QN meeting the axes in O. $OM=m$, etc. Then $mp=mq$, and the parabolae are

$$(x\pm y)^2-(m-p)x-(n-q)y-mp=0.$$

But $\quad l^2=-\Delta\div s^3$ where $s=(a+b-2h\cos\omega)/\sin^2\omega$.

Hence $\quad s=\sec^2\frac{\omega}{2}$ or $\operatorname{cosec}^2\frac{\omega}{2}$; $-\Delta=\frac{1}{4}[(m-p)\pm(n-q)]^2\operatorname{cosec}^2\omega$.

$$\therefore\ \text{product } l_1l_2=\tfrac{1}{8}[(m-p)^2-(n-q)^2]=\tfrac{1}{8}[MP^2 \backsim NQ^2]\sin\omega.$$

202. *A quadrilateral ABCD is inscribed to a circle, centre O, and circumscribed to a circle, centre I; OI cuts the circle ABCD in H and K. Show that* $AI^{-2}+CI^{-2}=HI^{-2}+KI^{-2}$. Trin. (C.), 1890.

Solution by G. Heppel and W. E. Jeffares.

$$AI^{-2}+CI^{-2}=r^{-2}\left(\sin^2\frac{A}{2}+\sin^2\frac{C}{2}\right)=r^{-2}\ (\text{for } A+C=\pi).$$

But $\quad HI^{-2}+KI^{-2}=r^{-2}$ (Casey, *Seq.*, 1888, p. 109); $\therefore$ etc.

[*v.* 5, 6, p. 188, M'Clelland, *Geometry of the Circle*, 1891.]

203. *Find the areal equations of the circles related to the triangle of reference.* St. John's (C.), 1895.

Solution by W. J. Greenstreet.

The areal equation is easily obtained by substituting $x=a\alpha$, etc., in the corresponding trilinear equation. Thus the equations of the various circles are:

circum-, $\quad \Sigma a^2yz=0$;

in-, $\quad \Sigma\sqrt{x(s-a)}=0$, or $\Sigma\cos\frac{A}{2}\sqrt{bcx}=0$;

ex-, $\cos\frac{A}{2}\sqrt{-bcx}+\sin\frac{B}{2}\sqrt{cay}+\sin\frac{C}{2}\sqrt{abz}=0$, etc.;

9 pt.-, $\Sigma x^2 bc\cos A-\Sigma a^2 yz=0$, and so on.

204. *If* $a+b+c+d+e+f+abc+def=0$,

and $ab+bc+ca=de+ef+fd$;

then $(d^2-1)(a+e)(a+f)=(a^2-1)(d+b)(d+c)$. King's (C.), 1895.

Solution by G. Heppel and Rev. H. P. Knapton.

$$e+f+def=-[a+b+c+d+abc],$$
$$d(e+f)+ef=ab+bc+ca;$$
$$\therefore\quad (d^2-1)(e+f)=a[1+d(b+c)+bc]+d+b+c+dbc,$$
$$(d^2-1)ef=-a[d+b+c+dbc]-d^2-d(b+c)-bc;$$
$$\therefore\quad (d^2-1)(a+e)(a+f)=(a^2-1)(d-b)(d-c).$$

205. *Show by means of the rule for finding the* H.C.F. *of two integral functions* $f(x)$ *and* $F(x)$ *which have no common divisor, that other integral functions* $\phi(x)$ *and* $\psi(x)$ *may be found such that* $f(x)\phi(x)+F(x)\psi(x)=1$ *identically. If* $f(x)=x^3+2x^2+2x+2$ *and* $F(x)=x^3-2x^2+2x-2$, *find* $\phi(x)$ *and* $\psi(x)$. *Utilize this to find a solution in positive integers of each of the equations,*

(i.) $187x^2-83y^2=1$, (ii.) $x^2-15521y^2=1$.

St. John's (C.), 1895.

Solution by F. S. Macaulay.

Let $F(x)$ be divided into $f(x)$ giving Q_1 for quotient and R_1 for remainder; then let R_1 be divided into $F(x)$ giving Q_2 for quotient and R_2 for remainder; and let the process be continued until a remainder R_n is obtained not containing x.

Then $f(x)=Q_1F(x)+R_1$, $F(x)=Q_2R_1+R_2$, $R_1=Q_3R_2+R_3$, etc.

These equations may be written,

$R_1=f(x)-Q_1F(x)$, $R_2=F(x)-Q_2R_1$, $R_3=R_1-Q_3R_2, \ldots$ $R_n=R_{n-2}-Q_nR_{n-1}$.

By substituting the value of R_1 given by the first of these in the second, then the values of R_1 and R_2 in the third, R_2 and R_3 in the fourth, and so on, we finally obtain the constant R_n in the form $\lambda_n f(x)+\mu_n F(x)$ where λ_n, μ_n are known integral functions of x. Dividing by R_n we have

$$f(x)\phi(x)+F(x)\psi(x)=1 \text{ identically.}$$

In the example given, applying the above process, we have

$$(x^3+2x^2+2x+2)(\tfrac{1}{2}x-1)^2-(x^3-2x^2+2x-2)(\tfrac{1}{2}x+1)^2=1.$$

If we put $x=5$ in this identity, we have

$$187\,.\,2^2-83\,.\,3^2=1.$$

Hence a solution of equation (i.) is $x=2$, $y=3$.

Again, squaring $187\,.\,2^2-83\,.\,3^2=1$,

we have $(187\,.\,2^2+83\,.\,3^2)^2-83\,.\,187\,.\,12^2=1$,

i.e. $1495^2-15521\,.\,12^2=1$.

Hence a solution of equation (ii.) is $x=1495$, $y=12$.

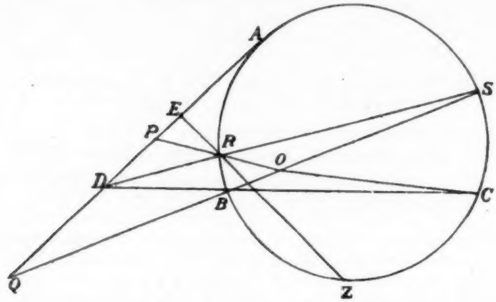
206. *The tangent at A to the circumcircle of a triangle ABC meets BC in D, and through D a variable chord DRS is drawn meeting the circle in R and S. If CR, BS meet DA in P, Q respectively, prove that the common chord of the circles ABC, PQR will pass through a fixed point in AD.* St. John's (C.), 1891.

Solution by R. F. Davis and G. Heppel.

Let *CR*, *BS* intersect in *O*. Then the polar of *D* passes through *O*, and the polar of *D* passes through *A*. Hence *OA* is the polar of *D*;

$\therefore$ $O\{ARDS\}$ is an harmonic pencil, and $\{APDQ\}$ is an harmonic range.

Take E the mid-point of AD. Then $EP.EQ=EA^2$ or ED^2. Produce ER to cut the circle in Z. Then $ER.EZ=EA^2=EP.EQ$;



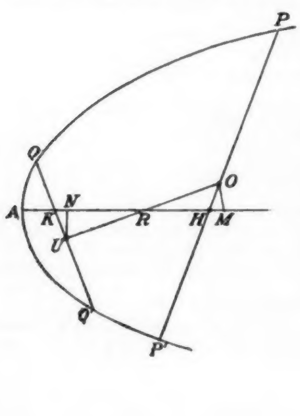
$\therefore$ RZ is the common chord of the circles ABC, PQR, and passes through a fixed point E.

207. *A heavy uniform bar, length b, is suspended in a horizontal position by two vertical strings each of length b. Find the couple which must be applied to keep the bar at rest in a horizontal plane, after it has been twisted through an angle of 60° round a vertical axis through its centre of gravity.* (O.)

Solution by J. Blaikie, E. P. Barrett, E. Fenwick, and others.

Since the bar has turned through 60°, the perpendicular from one extremity to the original position of the string at that extremity is $\frac{b}{2}$; therefore the string is at 30° to the vertical, and $T=\frac{W}{\sqrt{3}}$. Taking moments about the centre of gravity, we get for the moment of the couple $2.\frac{T}{2}.\frac{\sqrt{3}}{4}b$, or $\frac{Wb}{4}$.

208. *One of the common chords of a circle and a parabola passes through the centre of the circle; prove that the chords cut off on the axis a length equal to the latus rectum.* (O.)



Solution by R. F. Davis.

Let O, U be the mid-points of the common chords PP', QQ'; O the centre of the circle; OU perpendicular to QQ'.

PP', QQ' are equally inclined to the axis, therefore R the mid-point of OU lies on the axis, so that

$$OM=UN;\ RM=RN;\ HM=KN.$$

Then $HK=MN=2RN$

$=2$(sub-normal of vertex of diameter through U)

$=$latus-rectum.

209. *If* $(\sec\theta+\tan\theta)(\sec\phi+\tan\phi)(\sec\psi+\tan\psi)=1,$
then $\Sigma\sec^2\theta=1+2\Pi\sec\theta.$ (O.)

Solution by R. F. DAVIS.

Writing x, y, z for the secants, and α, β, γ for the tangents, and noting that $x^2-\alpha^2=1$, we have

$$\Pi(x+\alpha)=\Pi(x-\alpha)=1.$$

Eliminating α, β, γ we get

$$\Sigma x^2-1=2xyz.$$

Solution by E. FENWICK.

Writing $\sec\theta=\cosh\alpha$, etc., we have

$$\tan\theta=\sin h\alpha\,;$$

$$\therefore\quad \sec\theta+\tan\theta=\epsilon^{\alpha}, \text{ and similarly for the rest};$$

$$\therefore\quad \epsilon^{\alpha+\beta+\gamma}=1, \text{ or } \Sigma\alpha=0.$$

Whence $$\cosh\alpha=\cosh(\beta+\gamma),$$

or $$\sec\theta=\sec\phi\sec\psi+\tan\phi\tan\psi\,;$$

$$\therefore\quad (\sec\theta-\sec\phi\sec\psi)^2=(1-\sec^2\phi)(1-\sec^2\psi),$$

or $$\Sigma\sec^2\theta=1+2\Pi\sec\theta.$$

210. *If* $(z+y\cos A)/y\sin A=(x+z\cos B)/z\sin B=(y+x\cos C)/x\sin C,$ *and*

$$A+B+C=(2n+1)\pi,$$

each fraction equals Σ at A. (O.)

Solution by R. F. DAVIS, E. FENWICK, W. E. JEFFARES, C. SANDBERG, and others.

Writing $\frac{z}{y}\operatorname{cosec}A+\cot A=t$, etc.,

$$t^3-t^2\Sigma\cot A+t\Sigma\cot B\cot C-\Pi\cot A=\Pi\operatorname{cosec}A.$$

But $\Sigma\cot B\cot C=1$, and $\cot\omega=\Sigma\cot A=\Pi\operatorname{cosec}A+\Pi\cot A\,;$

$$\therefore\quad (t^2+1)(t-\cot\omega)=0, \text{ or } t=\cot\omega.$$

211. *Construct a triangle having its sides proportional to three integers, and the cosine of an angle equal to a given fraction, positive or negative.*

C. E. M'VICKER.

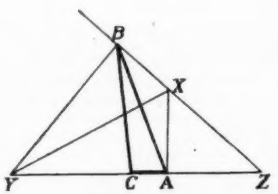


FIG. 1.

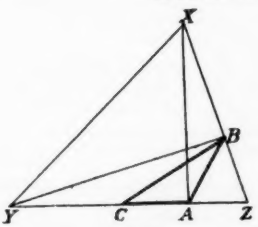


FIG. 2.

Solution by PROPOSER and G. HEPPEL.

Take a triangle XYZ, having X=supplement of given angle. Suppose $XY>XZ$, and commensurable with it. Bisect YZ in C, and draw XA, YB perpendicular to YZ, ZX.

ABC shall be the given triangle. For $A=\pi-X=$given angle, and

$$2YZ.BC=YZ^2, \quad 2YZ.CA=XY^2-XZ^2, \quad 2YZ.AB=2XY.BZ.$$

EXAMPLE. $\cos A=\frac{1}{3}$. Take $BX=1$, $XY=3$, $XZ=2$ (fig. 1).

Then $\qquad YZ^2=1, \quad XY^2-XZ^2=5, \quad 2XY.BZ=18;$

$$\therefore \quad a:b:c=17:5:18.$$

Again, $\cos A=-\frac{2}{3}$. Take $BX=4$, $XY=6$, $BZ=1$ (fig. 2), and we find

$$a:b:c=21:11:12.$$

It is worth noting that for $A=90°$, B and X coincide, and

$$a:b:c=XY^2+YZ^2:XY^2-XZ^2:2XY.XZ,$$

the well-known form for sides of right-angled triangles with sides proportional to integers.

212. *Solve the equation*

$$a_1-\frac{1}{a_2-}\frac{1}{a_3-}\dots-\frac{1}{a_n+}\frac{1}{x-}\frac{1}{a_n-}\dots-\frac{1}{a_1}=0. \quad \text{St. John's (C.), 1896.}$$

Solution by F. S. MACAULAY.

Let $\frac{p_r}{q_r}$ be the r^{th} convergent to $a_1-\frac{1}{a_2-}\frac{1}{a_3-}\dots$. Then it is known that

$$p_r=a_rp_{r-1}-p_{r-2},$$

$$\frac{p_r}{p_{r-1}}=a_r-\frac{p_{r-2}}{p_{r-1}},$$

$$\frac{p_{r-1}}{p_r}=\frac{1}{a_r-}\frac{p_{r-2}}{p_{r-1}}=\dots=\frac{1}{a_r-}\frac{1}{a_{r-1}-}\dots-\frac{1}{a_1}.$$

Hence the equation for n may be written

$$a_1-\frac{1}{a_2-}\dots-\frac{1}{a_n+}\frac{1}{x-}\frac{p_{n-1}}{p_n}=0,$$

or
$$a_1-\frac{1}{a_2-}\dots-\frac{1}{a_n+}\frac{p_n}{xp_n-p_{n-1}}=0$$

or
$$\frac{(xp_n-p_{n-1})p_n+p_np_{n-1}}{(xp_n-p_{n-1})q_n+p_nq_{n-1}}=0;$$

$$\therefore \quad x=0.$$

213. *The centre of gravity of a triangle is at the centre of a circumscribed ellipse. Show that ρ_A, ρ_B, ρ_C are proportional to a^3, b^3, c^3 respectively; and that $\rho_A\rho_B\rho_C=R^3$, where ρ_A, etc., are radii of curvature at A, etc., and R is the circum-radius of the triangle.* Trin. (C.), 1896.

Solution by G. HEPPEL.

Let A, B, C be $(p\cos\beta, q\sin\beta)$, (γ), (δ).

By hypothesis $\qquad \Sigma\cos\beta=\Sigma\sin\beta=0,$

whence we get

$$\cos\gamma-\cos\delta=\sqrt{3}\sin\beta, \quad \sin\gamma-\sin\delta=\sqrt{3}\cos\beta, \quad \Sigma\sin^2\beta=\tfrac{3}{2}.$$

Now $\qquad 2\Delta/pq=\Sigma\sin\beta(\cos\gamma-\cos\delta)=\frac{3}{2}\sqrt{3};$

$$\therefore \quad 4\Delta=3\sqrt{3}pq;$$

so that
$$\rho_A=(p^2\sin^2\beta+q^2\cos^2\beta)^{\frac{3}{2}}/(pq),$$
$$a^2=p^2(\cos\gamma-\cos\delta)^2+q^2(\sin\gamma-\sin\delta)^2;$$
$$\therefore\quad \rho_A=a^3/3\sqrt{3}pq=a^3/4\Delta,$$
and
$$\rho_A\rho_B\rho_C=R^3.$$

Solution by W. J. GREENSTREET.

The ellipse and triangle ABC are the projections of a circle and an equilateral triangle $A'B'C'$. If l, m, n are the diameters of the ellipse parallel to BC, etc., and p, q the semi-axes, we have $\frac{B'C'}{BC}=\frac{p}{l}$, etc.

But
$$B'C'=C'A'=A'B';$$
$$\therefore\quad a:b:c=l:m:n.$$

Also
$$\rho_A=\frac{l^3}{pq},\text{ etc., and } R=\frac{lmn}{pq};$$
$$\therefore\quad \rho_A/l^3=\rho_B/m^3=\rho_C/n^3,\text{ i.e. } \rho_A/a^3=,\text{ etc.}$$

and
$$\rho_A\rho_B\rho_C=\frac{l^3m^3n^3}{p^3q^3}=R^3.$$

Another solution has been received from C. SANDBERG.

REVIEWS AND NOTICES.

An Elementary Course of Infinitesimal Calculus. By PROFESSOR HORACE LAMB, M.A., F.R.S., formerly Fellow of Trinity College, Cambridge. (Cambridge University Press. 8vo, pp. xx. + 616.)

The Differential Calculus in England seems to have a fatal habit of losing sight of the Calculus on the continent. The work of the Cambridge Analytical Society is a curious chapter in the history of Mathematics; by its efforts, in 1817-20, our insularity was done away with for a time. But the seventy years since then have seen another relapse; and we have now the pleasure of recording the symptoms of another recovery.

A sharp criticism of our backwardness in this subject was written in 1892 by Miss C. A. Scott for the *Bulletin* of the New York Mathematical Society, on the appearance of a fresh edition of Edward's *Differential Calculus*. The charges of the critics may briefly be stated as follows:

Firstly, no notice has been taken of the development of analysis on the continent, except when it has proceeded on the old lines. No attempt has been made to rewrite the text-books in the light furnished by the modern theories of functions, both of a real variable and of a complex variable.

Secondly, the foundations of the subject are badly laid; in particular, in work relating entirely to the properties of functions, no definite meaning is attached to the word function itself.

Thirdly, the definitions of the terms and the enunciations and proofs of the fundamental theorems have, in many cases, long been disused outside England, as either incomplete or erroneous.

Fourthly, the larger treatises wander into the theory of higher plane curves, thereby poaching on another subject, and crowding out the Calculus itself.

As an example of the difficulties ignored in the average text-book, let us take Lagrange's theorem.

If
$$y=z+x\phi(y),\ \dots\dots\dots\dots(1)$$
then, by Lagrange's theorem, y can be expanded in the form
$$y=z+x\phi(z)+\frac{x^2}{2!}\frac{d}{dz}\{[\phi(z)]^2\}+\dots+\frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\{[\phi(z)]^n\}+\dots.$$

Now, the last equation gives *y uniquely* in terms of x and z, if $\phi(y)$ be a uniform function. But equation (1) will in general have an infinite number of roots y, if x and z are given. Which one has survived, and what has become of the others?

Doubtless he would be a bold man who would introduce the elements of the theory of functions of a complex variable into a treatise on the Infinitesimal Calculus; but it is easier than much of the matter now needlessly included; and it is hard to see how a reasonable account of expansion theorems can be written without it.

Interest in questions of this nature may be said to date from the appearance of Professor Forsyth's *Theory of Functions* in 1893. The effect is now being distinctly felt. In some recent English books, we find alterations in the order of the chapters, omission of old paragraphs, insertion of new ones, and long apologetic prefaces—the signs of a general state of transition. The Calculus is even now in the melting-pot.

Professor Lamb's new book is not an exhaustive treatise, and is not even designed for the professional mathematician at all. But its publication is nevertheless a notable event in the history of our mathematical teaching.

It is not a large book; it meets the beginner at the beginning, and leaves him after considering Taylor's theorem. But importance is not to be measured by size alone.

The most notable differences from the old order are:

1. The elements of the Differential and Integral Calculus are developed side by side.

2. The theory of functions of a real variable has been extensively borrowed from.

3. Applications to maxima and minima, geometrical questions, etc., occur immediately after the chapters on Differential Coefficents. Taylor's theorem is deferred until the last chapter of the book.

4. The theories of indeterminate forms, algebraic curves, and functions of more than one variable, are almost completely left out.

These points are well worthy of being considered in order. The development of differentiation and integration together is now very generally accepted by the mathematical world. The exercises in each help to fix the other in the beginner's mind; the proofs of many theorems given in books on the Differential can be simplified by the elementary theorems of the Integral Calculus. But above all, the separation of the Calculus into two distinct subjects is unscientific in theory, and injurious to its development in practice.

The second point we have noticed—the inclusion of investigations from the theory of functions of a real variable—raises a far more debatable question. The old text-books ignored alike the modern theories of functions of a real variable (as given *e.g.* in Dini) and the theory of functions of a complex variable (as given *e.g.* in Forsyth). The impulse now is to include something of them. But what shall we include? The theory of functions of a complex variable, in its present state of development, considers only those functions which are 'monogenic,' *i.e.* which can generally be expressed as power-series. In thus restricting the nature of the function, however, no restriction is placed on the range of values of the independent variable: it may take any complex value.

The treatment of such functions from the point of view of the Differential Calculus is simple; at any ordinary point they possess derivates of all orders.

The theory of functions of a real variable, on the other hand, considers more general functions—such, for example, as can be expressed by Fourier's series. But the independent variable is restricted to take only real values.

Such functions cannot necessarily be represented by power-series; they do not necessarily possess derivates of any or all orders; they may even have two different derivates for the same value of the variable—the 'progressive' and the 'regressive.' Professor Lamb has chosen, in his book on the Calculus, to borrow from this theory, and to attach this wide meaning to the word 'function.' "One variable quantity," he says, "is said to be a function of another when, other things remaining the same, if the value of the latter be fixed, that of the former becomes determinate."

We cannot help feeling that he has made a mistake. Such a function does not in general possess a derivate, and the methods Professor Lamb develops do not apply

to it at all. The fact is, the Differential Calculus is a machine, and is constructed to deal with functions which can be represented by power-series. It is useful for more general functions only in so far as, near certain values of the independent variable, they are so expressible. In a book which aims at developing a method, it is surely best to restrict the subjects of operation to those for which the method applies.

This last suggestion has actually been carried out by Méray, in his *Analyse Infinitésimal*. All that is needed is to restrict the definition of functionality, so as to make it include only monogenic functions; and then, without any loss of rigour, the whole theory of progressive and regressive differential coefficients, functions with finite discontinuities, etc., is swept out of the Calculus. The subject is made as easy as in the good old days when rigour was unknown.

Indeed, the difficulty and dulness of the elementary theory of functions of a real variable ought alone to secure its banishment from an elementary text-book. It is an assemblage of quantities ϵ whose moduli are less than positive quantities η, chosen arbitrarily as small as we please.

The third point we have noticed,—the early introduction of geometrical and other applications,—will probably meet with general approval. It is without doubt good to connect the new subject at its very beginning with easily understood problems like these.

In a book on Professor Lamb's plan, Taylor's theorem can quite well be deferred to the last chapter, or, indeed, left out altogether, for nothing else is made to depend on it. In such a book as Méray's, it takes its place at the beginning, as the essence of the Calculus; in fact, Méray *defines* the successive differential coefficients, not as limiting values of ratios, but as the coefficients in the Taylor expansion.

The omission of the theory of higher plane curves was one of the reforms advocated by Miss Scott in the *New York Bulletin*; the omission of the theories of indeterminate forms, and functions of more than one variable, are justified by the size and scope of the work.

We think that, if anything, Professor Lamb is too indefinite as to the class of readers he is writing for. If the book is intended for engineers, they will be chilled to find in the first chapter a discussion on the limits of assemblages; if it is meant for mathematicians, the work on moments of inertia and homogenous strain might well be left to other books. Reviewing the book as a whole, we heartily wish it success. It includes seven chapters of what would commonly be called Differential Calculus, four of Integral Calculus, two of Differential Equations, and a good chapter on the much-neglected but enormously important theory of Infinite Series. There are plenty of good figures and good examples; and teachers will probably find it the best guide to give to those who are entering for the first time the temple of the higher mathematics. E. T. W.

Theoretical Mechanics, an Introductory Treatise on the Principles of Dynamics, with Applications and Numerous Examples, by A. E. H. Love, M.A., F.R.S., Fellow and Lecturer of St. John's College, Cambridge. (Camb. Univ. Press.)

The treatise before us is among the most interesting text-books that have appeared in recent years; like the same author's work on *Elasticity* it is conspicuous for the thoroughness of its treatment of fundamental principles, as well as for the excellence of its style.

The book commences with an exposition of the geometrical properties of vectors: the parallelogram of velocities, with its difficult initial conception of a point moving with two independent velocities, is here replaced by the corresponding theorem on relative motion. Then follows (Chap. IV.) an analytical treatment of central orbits under given accelerations. After this preliminary kinematics we have a discussion of the principles of dynamics. Since absolute rest is incapable of realization, and it is only relative velocities and accelerations which can be determined, it is clear that the Newtonian measure of a force in terms of the acceleration that it produces in a body of known mass is inadequate. For the old Laws of Motion there are here substituted a number of postulates and definitions applicable to ideal bodies; in these the accelerations are referred to an arbitrarily selected 'frame' or set of axes, and the logical deductions form the science of 'rational mechanics.' By choosing the frame in a suitable manner we may obtain the motion of natural bodies.

In Chapters VI., VII., VIII., the discussions of such general principles as that of D'Alembert, the transmissibility of force, and the conservation of energy, are admirable in their lucidity and completeness.

Chapters IX., X. deal with the ordinary problems of particle dynamics.

The motion of a rigid body in two dimensions is next treated. Perhaps the chief defect of existing text-books is their failure to lay adequate stress on the legitimacy of the equation obtained by taking moments about any point whatever, even if a body there have acceleration: in other words, about a fixed point with which the moving point instantaneously coincides. We are glad to see that the present work clearly states the equivalence of the 'kinetic reaction' to the externally applied forces (§§ 219, 221), and makes use of the general equation of moments on p. 241. We think, however, that a warning against the common error of regarding $m(r^2+k^2)\dot{\omega}$ as the effective couple about the instantaneous centre might with advantage be put into the text. A further possible addition would be a statement of the method of virtual work applied to the statical problem to which D'Alembert's principle reduces a problem in dynamics.

Chapter XII. contains investigations of collision and impulsive motion, of initial motions and small oscillations, followed by the discussion of problems connected with a chain.

The only error of any importance that we have noticed occurs in Ex. 5, on page 174. In explaining the mode in which friction is exerted when a locomotive moves a train, the author says that whenever steam is 'on,' even at starting, the driving wheel 'skids,' and there is sliding friction. He says further, "The resistance of the rails arises from rolling friction and the pull of the engine from sliding friction, and, since the ratio of rolling friction and pressure at a point of contact is always less . . . than that of sliding friction to pressure, there is no difficulty in seeing how a train can be set in motion by a locomotive of smaller mass than the train." It is, we believe, the universal practice of engine-drivers to reduce steam at once if the driving-wheels should happen to 'skid' and fly round; the pull on the train is also perceptibly less when the engine so 'runs away,' than when the driving wheels are rolling. The fact too, that the train would be pulled along if the coach-wheels were either perfectly smooth or were cog-wheels moving on rackwork rails, suggests that the explanation above given is not satisfactory.

In addition to scattered groups of straightforward examples, there are five collections of problems, including, in all, upwards of 660; these cannot fail to be of immense value.

To those who wish for a satisfactory foundation to their theory as well as for elegance and power in their methods, Mr. Love's book is most heartily to be recommended. G. T. W.

Famous Problems of Elementary Geometry. An authorized translation of PROFESSOR F. KLEIN'S *Vorträge über ausgewählte Fragen der Elementargeometrie.* By PROFESSOR W. W. BEMAN and PROFESSOR D. E. SMITH. (Ginn & Co.) Of continental mathematics none is better known to English readers than the work of Professor Klein. The present unpretending volume may be regarded as an elementary sequel to the famous Chicago "Lectures" of 1893. Its origin was a short course of lectures delivered at Göttingen in 1894, which were subsequently published and presented by the author to the German Association for the Advancement of the Teaching of Mathematics and the Natural Sciences. Needless to say the volume is of great interest from beginning to end. The first chapter, in some respects the most difficult in the book, is devoted to proving the theorem that *any quantity which can be expressed in terms of integers and square roots* (*including square roots under square roots, etc.*) *is a root of an irreducible equation whose degree is a power of* 2. Now the measure of any quantity constructed by means of straight lines and circles is always expressible in terms of integers and square roots; and it consequently follows that a quantity which is a root of an irreducible equation, whose degree is *not* a power of 2, *cannot* be constructed by means of straight lines and circles, that is, by ruler and compass. This proves the impossibility of solving by such means alone two of the great problems of antiquity, viz., the duplication of the cube and the trisection of an angle, both of which depend on the solution of irreducible equations of the third degree. Notwithstanding its long-proved impossibility, the problem of the trisection of an angle seems to be

as fascinating, and to provoke as many attempts at solution, as ever. Only within the last few weeks another hopeless attack on the problem has reached us, the writer submitting it in the confident expectation of winning a prize, which, he understands, has been offered in London for a solution! The third chapter of the book is devoted to showing what regular polygons can be constructed under the same restrictions as before; and this is followed by a complete construction of the regular 17-sided polygon.

The latter part of the book is on transcendental numbers, and is, if possible, still more interesting than the earlier part, since it deals with more general ideas. Numbers may be divided into two classes, viz., *algebraic* numbers, or all numbers which are roots of algebraic equations with integral coefficients, and *transcendental* numbers, or numbers which do not satisfy any algebraic equation. The fact of the existence of transcendental numbers has been recognized and proved for more than half a century. What is proved in the book on this subject is summed up (p. 76) in the concise and general theorem that *in an equation of the form*

$$a + be^k + ce^l + \dots = 0$$

the exponents k, l ... and coefficients a, b, c ... cannot all be algebraic numbers. As particular cases we may take the equations $a - e = 0$, $1 + e^{i\pi} = 0$, $y - e^x = 0$ or $x = \log y$, which show respectively that e, π, and the logarithms of algebraic numbers are all transcendental. Similarly for the circular and inverse circular functions, etc., etc. We must, however, remark that the so-called elementary proof on p. 54 of the existence of transcendental numbers seems to us no proof at all, and so palpably fallacious as not to need refutation.

Two points touched upon in the book are of primary importance. One (p. 61) is the distinction between practical and theoretical convergence. The number 10^{100}, for example, is so great as to be altogether beyond conception; but theoretically it is finite, and inconceivably small in comparison with other finite numbers. So again the divergence of the series $1 + \frac{1}{2} + \frac{1}{3} + \dots$ is a fundamental fact in theory, yet if taken to 10^{100} terms the sum is less than 231. The other point (pp. 50, 51) refers to the comparison of absolute infinities. "The proposition that the part is greater than the whole is not true for infinite masses." Absolute infinities cannot be compared in respect to magnitude. It is not correct to assume that the four infinite quadrants into which a plane is divided by two perpendicular lines are equal to one another. "The aggregate of real algebraic numbers and the aggregate of positive integers can be brought into a one-to-one correspondence," notwithstanding the fact that between any two successive integers an infinite number of real algebraic numbers exist. F. S. M.

Elementary Geometrical Statics. By W. J. Dobbs, M.A. (Macmillan, pp. 340.) This book is intended as an introduction to Graphic Statics, "preparing the way for such works as Clarke's *Graphic Statics* or Hoskin's *Elements of Graphic Statics.*" It will undoubtedly be welcome to those at, or about to enter, the "shop," but meagre attention having been hitherto paid in the ordinary textbooks to what is so important to the engineer. The examples are numerous and well-selected; the majority seem to be original, and the residue to be from recent Woolwich papers. The space and force diagrams are beautifully drawn, and generally exhibited side by side. The chapters on systems of rods and stiff frameworks are extremely well done. To use a hackneyed expression, this volume certainly "fills a gap." W. J. G.

Solutions of the Exercises in Taylor's Euclid. Books VI., XI. By W. W. Taylor, M.A. (Cambridge University Press.) This book of solutions may prove useful to those who attack the harder exercises in the Pitt Press (Taylor's) Euclid. Besides the solutions the book contains some slight additions, such as the proof of the converse of Casey's theorem in regard to four circles which all touch the same fifth circle, and a construction for the centre of a given circle by using the compass only. F. S. M.

ANNUAL MEETING.

The Annual Meeting of the Mathematical Association was held at University College, Gower Street, W.C., on Saturday, 15th January, 1898. There were 24 members present.

The Report of the Council for 1897 was read and adopted, and the elections of 46 new members of the Association were confirmed. The Rules, which had been revised, were discussed and passed, and the Treasurer's audited financial statement was accepted. After officers and council for 1898 had been elected, a paper on "Some Curiosities in Division" was read by E. M. Langley, M.A., and a paper by Prof. H. W. Lloyd Tanner on "A Class of Algebraic Functions" was, in the absence of the author, read by the President. In consequence of the lateness of the hour the other papers were postponed.

The next meeting of the Association will be held at University College, *at Eight p.m., on Friday, 6th May.*

BOOKS, MAGAZINES, ETC., RECEIVED.

The Tutorial Chemistry. Part II., Metals. By G. H. Bailey, D.Sc., Ph.D. (W. B. Clive.)

Further Development of the Relations between Impulsive Screws and Instantaneous Screws. 11th and 12th (concluding) Memoirs on the "Theory of Screws." By Sir Robert S. Ball, LL.D., F.R.S. ("Transactions of the Royal Irish Academy," vol. xxxi., pp. 99-196.)

Contributions to the Geometry of the Triangle. By R. J. Aley, A.M. ("Publications of the University of Pennsylvania; Mathematics, No. 1.")

On an Extension of the Wallace Problem. By Dr. N. Quint. ("Nieuw Archief voor Wiskunde.")

Sui gruppi continui di trasformazioni cremoniane dello spazio. By Professors Federigo Enriques and Gino Fano. ("Annali di Matematica.")

Risoluzione delle singolarità puntuali delle superficie algebriche. By Professor Beppo Levi. ("Atti della R. Accademia delle Scienze di Torino," vol. xxxiii.)

Teoria dei limiti. By Professor R. Bettazzi. ("Rivista di Matematica.")

The American Mathematical Monthly. August to December, 1897. Edited by Professor B. F. Finkel and J. M. Colaw, A.M. (A. Hermann, 8 Rue de la Sorbonne, Paris.)

Journal de Mathématiques Élémentaires. October, 1897, to February, 1898. Edited by Prof. G. Mariaud. (Librairie Ch. Delagrave, 15 Rue Soufflot, Paris.).

Periodico di Matematica. November, 1897, to February, 1898. Edited by Prof. Giulio Lazzeri. (Tipografia di Raffaello Giusti, Livorno.)

Bolletino della Associazione Mathesis. Anno II., Nos. 1-3.

Bulletin de la Société Physico-Mathématique de Kasan. Vol. vii., Nos. 2, 3.

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